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BENDING AND BUCKLING OF RECTANGULAR SANDWICH PLATES

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SUMMARY

Differential equations and boundary conditions are derived for the bending and buckling of sandwich plates. The buckling load is calculated for a simply supported plate subjected to edgewise compression. The formulas obtained are evaluated numerically and the results are plotted in a diagram. The theory is in satisfactory agreement with results of tests carried out at the Forest Products Laboratory.

INTRODUCTION

The expression "sandwich plate" designates a composite plate consisting of two thin faces and a thick core. In airplane construction the faces are usually of aluminum alloy and the core is of some lightweight material such as an expanded plastic or balsa wood. In the latter case the fibers of the wood are in general arranged perpendicularly to the plane of the plate. Since thus the modulus of elasticity of the core in the plane of the plate is of the order of magnitude of one-thousandth of that of the faces, the normal stresses in the core are of little importance in resisting bending moments even though the usual ratio of face thickness to core thickness is between one-tenth and one-hundredth. On the other hand the core performs a task in transmitting shear forces and undergoes considerable shearing deformations because its modulus of shear is low. Hence shearing deformations must not be disregarded in the analysis of sandwich plates.

In an earlier paper (reference 1) the differential equations of bending were derived by means of the principle of virtual displacements for sandwich beams subjected to transverse and axial loads. Integration of the equations yielded formulas for buckling load and deflection which were found to be in good agreement with test results. For this reason in the present paper the bending and the buckling of sandwich plates are analyzed on the basis of the same assumptions as those underlying the earlier investigation.

Sandwich plates have already been discussed by various authors. In 1942, Leggett and Hopkins (reference 2) gave a rigorous and an approximate solution of the problem of the buckling of sandwich plates simply supported along the four edges. In 1943, Van der Neut (reference 3) carried out a rigorous analysis of the same problem. In 1945, March and Smith (reference 4) presented approximate strain-energy solutions for various edge conditions. In 1946, Bijlaard (reference 5) proposed a simple procedure which gives rigorous results for simply supported plates and approximate ones for other edge conditions. In 1948, Reissner (reference 6) developed a large-deflection theory for sandwich plates and in the same year Libove and Batdorf (reference 7) presented a new small-deflection theory.

In spite of this abundance of theoretical work in the field, the development of the present theory is justified in the author's opinion because it is simple enough to permit the solution of the problem of the buckling of rectangular sandwich plates having various edge conditions. The solution by means of the rigorous theories cited presents great difficulties in cases when any of the edges is not simply supported. At the same time the present theory is more reliable than those which have already been used to obtain results for the less-simple edge conditions.

This report contains the development of the theory and a solution of the buckling problem for the case when all the four edges are simply supported. It is hoped that the graph containing the numerical results will be found convenient to use in the industry.

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SYMBOLS

B	integration constant
c	core thickness, inches
C	multiplier in buckling-stress expression; integration constant
D	bending rigidity of plate, pound-inches squared per inch
D_f	bending rigidity of two independent faces, pound-inches squared per inch

D_o	bending rigidity of sandwich panel, neglecting rigidity of independent faces and of core, pound-inches squared per inch
E	Young's modulus, psi
F	form factor
G	shear modulus, psi
K	side ratio of bulge
L_x,L_y	edge lengths of sandwich panel, inches
n	number of half waves in direction of load
P_{cr}	buckling load, pounds per inch
P_f	one-quarter buckling load corresponding to D_f, pounds per inch
P_o	one-quarter buckling load corresponding to D_o, pounds per inch
P_x	compressive edge load in x-direction, pounds per inch
P_y	compressive edge load in y-direction, pounds per inch
q	distributed transverse load, psi
R	sandwich buckling parameter $\left(G_c/F\sigma_{cr,f}\right)$
S	integration constant; refers to surface area during integration
t	face thickness, inches
u	x-displacement, in opposite directions in the two faces, inches
U	strain energy, inch-pounds
v	y-displacement, in opposite directions in the two faces, inches
V	potential of external load
w	z-displacement, inches

x	rectangular coordinate in plane of faces, inches
y	rectangular coordinate in plane of faces, inches
z	rectangular coordinate perpendicular to plane of faces, inches
γ	shear strain
δ	variation sign
Δ^2	Laplace operator
ϵ	normal strain
μ	Poisson's ratio
σ	normal stress, psi
σ_{cr}	buckling stress of sandwich panel, psi
$\sigma_{cr,f}$	buckling stress of two independent faces, psi
τ	shear stress, psi

Subscripts:

b	bending
c	core
f	face
q	transverse load
s	shear

DERIVATION OF DIFFERENTIAL EQUATIONS

The differential equations of the problem are derived by means of the principle of virtual displacements from the essential parts of the strain energy stored in the sandwich plate. The strain-energy quantities considered as essential are those caused in the faces by extensions in the x- and y- directions (see figs. 1 and 2 for the notation) and by shear in the xy-plane; the strain energy of bending in the faces; and the strain energy of shear caused in the core by angular changes in the xz- and yz-planes. Consequently the stresses in the xy-plane in the core are assumed to contribute only negligible amounts to the total strain energy. This assumption is justifiable when the moduli E and G of the core are small as compared with those of the faces. Moreover, normal strains in the core in the z-direction are disregarded. The earlier investigations of the sandwich beam proved this procedure to be satisfactory. Finally, the strain energy stored in the faces because of shear perpendicular to the faces is neglected. This is permissible just as the shear strain energy stored in a beam subjected to bending can be disregarded provided the beam is long enough. In the case of the sandwich plate the ratio of the length or width of the plate to the thickness of a face is always large.

The deformations are described by means of three functions u, v, and w of the coordinates x and y (see figs. 3 and 4). The function u represents a displacement in the positive x-direction in the upper face and a simultaneous displacement of equal magnitude in the negative x-direction in the lower face. The definition of the v-displacements is obtained from the preceding one through replacing x by y. During the u- and v-displacements the faces, and the entire sandwich plate, remain plane. The plate becomes curved during the w-displacements which take place in the z-direction through shearing the core. The w-displacements do not cause force resultants (corresponding to membrane stresses) in the x- and y-directions in the individual faces but they give rise to bending and twisting moments in them because of the nonvanishing bending and torsional rigidity of each face. With

$$\sigma_x = \frac{E}{1 - \mu^2}\left(\frac{du}{dx} + \mu \frac{dv}{dy}\right) \qquad \epsilon_x = \frac{du}{dx}$$

$$\sigma_y = \frac{E}{1 - \mu^2}\left(\mu \frac{du}{dx} + \frac{dv}{dy}\right) \qquad \epsilon_y = \frac{dv}{dy}$$

$$\tau_{xy} = G\left(\frac{du}{dy} + \frac{dv}{dx}\right) \qquad \gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

the strain energy stored in one face

$$U_f = (1/2)t \iint \left(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy}\right)dx\ dy$$

becomes

$$U_f = (1/2)\left[Et/\left(1 - \mu^2\right)\right]\iint\left[\left(u_x^2 + 2\mu u_x v_y + v_y^2\right) + (1/2)(1 - \mu)\left(u_y + v_x\right)^2\right]dx\ dy \tag{1}$$

where the subscripts x and y denote differentiation with respect to x and y, respectively.

The strain energy of bending in one face plate can be calculated from the known formula (see, for instance, equation (48) on p. 50 of reference 8)

$$U_b = (D/2)\iint_S\left[\left(w_{xx}^2 + 2\mu w_{xx}w_{yy} + w_{yy}^2\right) + 2(1 - \mu)w_{xy}^2\right]dx\ dy \tag{2}$$

where D is the bending rigidity of the plate. In the case of a face

$$D = Et^3/\left[12\left(1 - \mu^2\right)\right] \tag{2a}$$

The angles of shear in the core are

$$\gamma_{xz} = \left[2u/(c + t)\right] - w_x$$

$$\gamma_{yz} = \left[2v/(c + t)\right] - w_y$$

Hence the strain energy of shear stored in the core is

$$U_s = \left(G_c c/2\right)\iint_S\left[\left(\frac{2u}{c + t} - w_x\right)^2 + \left(\frac{2v}{c + t} - w_y\right)^2\right]dx\ dy \tag{3}$$

where G_c is the shear modulus of the core.

The potential of the distributed transverse load q is

$$V_q = -\iint_S qw\ dx\ dy \tag{4}$$

while the potential of the distributed compressive loads P_x and P_y is

$$V_c = -(P_x/2)\iint_S w_x{}^2 dx\, dy - (P_y/2)\iint_S w_y{}^2 dx\, dy \tag{5}$$

According to the principle of virtual displacements

$$\delta(U + V) = 2\delta U_f + 2\delta U_b + \delta U_s + \delta V_q + \delta V_c = 0 \tag{6}$$

Substitutions yield

$$\begin{aligned}
\delta(U + V) = {} & \left[Et/\left(1 - \mu^2\right)\right] \iint_S \Big\{2\left(u_x\delta u_x + \mu u_x\delta v_y + \mu v_y\delta u_x + v_y\delta v_y\right) + \\
& (1 - \mu)\left[\left(u_y + v_x\right)\left(\delta u_y + \delta v_x\right)\right]\Big\}\, dx\, dy + \\
& (1/12)\left[Et^3/\left(1 - \mu^2\right)\right] \iint_S 2\Big[w_{xx}\delta w_{xx} + \mu w_{xx}\delta w_{yy} + \mu w_{yy}\delta w_{xx} + \\
& w_{yy}\delta w_{yy} + 2(1 - \mu)w_{xy}\delta w_{xy}\Big] dx\, dy + \\
& G_c c \iint_S \left[\left(\frac{2u}{c + t} - w_x\right)\left(\frac{2\delta u}{c + t} - \delta w_x\right) + \right. \\
& \left.\left(\frac{2v}{c + t} - w_y\right)\left(\frac{2\delta v}{c + t} - \delta w_y\right)\right] dx\, dy - \\
& \iint_S q\delta w\, dx\, dy - P_x \iint_S w_x\delta w_x\, dx\, dy - P_y \iint_S w_y\delta w_y\, dx\, dy = 0
\end{aligned} \tag{7}$$

The derivatives of the variations of the unknown functions u, v, and w can be eliminated if use is made of Gauss' theorem. In the case of the term $u_x \delta u_x \, dx \, dy$ of equation (7) the transformations are:

$$\iint_S u_x \delta u_x \, dx \, dy = \iint_S \left[\left(u_x \delta u \right)_x - u_{xx} \delta u \right] dx \, dy$$

$$= \oint u_x \delta u \, dy - \iint_S u_{xx} \delta u \, dx \, dy$$

where the line integral is extended around the entire boundary. In a similar manner one obtains

$$\iint v_y \delta u_x \, dx \, dy = \oint v_y \delta u \, dy - \iint v_{xy} \delta u \, dx \, dy$$

$$\iint u_x \delta v_y \, dx \, dy = -\oint u_x \delta v \, dx - \iint u_{xy} \delta v \, dx \, dy$$

$$\iint v_y \delta v_y \, dx \, dy = -\oint v_y \delta v \, dx - \iint v_{yy} \delta v \, dx \, dy$$

$$\iint u_y \delta u_y \, dx \, dy = -\oint u_y \delta u \, dx - \iint u_{yy} \delta u \, dx \, dy$$

$$\iint v_x \delta u_y \, dx \, dy = -\oint v_x \delta u \, dx - \iint v_{xy} \delta u \, dx \, dy$$

$$\iint u_y \delta v_x \, dx \, dy = \oint u_y \delta v \, dy - \iint u_{xy} \delta v \, dx \, dy$$

$$\iint v_x \delta v_x \, dx \, dy = \oint v_x \delta v \, dy - \iint v_{xx} \delta v \, dx \, dy$$

$$\iint u \delta w_x \, dx \, dy = \oint u \delta w \, dy - \iint u_x \delta w \, dx \, dy$$

$$\iint w_x \delta w_x \, dx \, dy = \oint w_x \delta w \, dy - \iint w_{xx} \delta w \, dx \, dy$$

$$\iint v \delta w_y \, dx \, dy = -\oint v \delta w \, dx - \iint v_y \delta w \, dx \, dy$$

$$\iint w_y \delta w_y \, dx \, dy = -\oint w_y \delta w \, dx - \iint w_{yy} \delta w \, dx \, dy$$

In the case of the terms multiplied by $Et^3/\left[12\left(1 - \mu^2\right)\right]$, Gauss' theorem has to be used twice in succession. Thus for the term $w_{xx}\delta w_{xx}$ dx dy the transformations are:

$$\iint w_{xx}\delta w_{xx}\,dx\,dy = \iint \left[\left(w_{xx}\delta w_x\right)_x - w_{xxx}\delta w_x\right]dx\,dy$$

$$= \oint w_{xx}\delta w_x\,dy - \iint w_{xxx}\delta w_x\,dx\,dy$$

$$= \oint w_{xx}\delta w_x\,dy - \iint \left[\left(w_{xxx}\delta w\right)_x - w_{xxxx}\delta w\right]dx\,dy$$

$$= \oint w_{xx}\delta w_x\,dy - \oint w_{xxx}\delta w\,dy + \iint w_{xxxx}\delta w\,dx\,dy$$

In a similar manner one obtains:

$$\iint w_{xx}\delta w_{yy}\,dx\,dy = \iint w_{xxyy}\delta w\,dx\,dy + \oint w_{xxy}\delta w\,dx - \oint w_{xx}\delta w_y\,dx$$

$$\iint w_{yy}\delta w_{xx}\,dx\,dy = \iint w_{xxyy}\delta w\,dx\,dy - \oint w_{xyy}\delta w\,dy + \oint w_{yy}\delta w_x\,dy$$

$$\iint w_{yy}\delta w_{yy}\,dx\,dy = \iint w_{yyyy}\delta w\,dx\,dy + \oint w_{yyy}\delta w\,dx - \oint w_{yy}\delta w_y\,dx$$

$$\iint w_{xy}\delta w_{xy}\,dx\,dy = \iint w_{xxyy}\delta w\,dx\,dy - \oint w_{xyy}\delta w\,dy - \oint w_{xy}\delta w_x\,dx$$

$$= \iint w_{xxyy}\delta w\,dx\,dy + \oint w_{xxy}\delta w\,dx + \oint w_{xy}\delta w_y\,dy$$

Finally the line integral

$$\oint w_{xy}\left(\delta w_x\,dx - \delta w_y\,dy\right)$$

becomes after integrations by parts:

$$\oint w_{xy}(\delta w_x\,dx - \delta w_y\,dy) = 2\left[-\left(w_{xy}\delta w\right)_{\substack{x=0\\y=0}} + \left(w_{xy}\delta w\right)_{\substack{x=L_x\\y=0}} - \left(w_{xy}\delta w\right)_{\substack{x=L_x\\y=L_y}} + \left(w_{xy}\delta w\right)_{\substack{x=0\\y=L_y}}\right] - \oint w_{xxy}\delta w\,dx + w_{xyy}\delta w\,dy$$

Substitutions in equation (7) yield

$$\begin{aligned}
\delta(U + V) = \iint\Bigg(&\left\{\frac{-Et}{1-\mu^2}\left[2u_{xx} + (1-\mu)u_{yy} + (1+\mu)v_{xy}\right] + \frac{2G_c c}{c+t}\left[\left(\frac{2}{c+t}\right)u - w_x\right]\right\}\delta u + \\
&\left\{\frac{-Et}{1-\mu^2}\left[2v_{yy} + (1-\mu)v_{xx} + (1+\mu)u_{xy}\right] + \frac{2G_c c}{c+t}\left[\left(\frac{2}{c+t}\right)v - w_y\right]\right\}\delta v + \\
&\left\{\frac{Et^3}{6(1-\mu^2)}\Delta^4 w + G_c c\left[\left(\frac{2}{c+t}\right)(u_x + v_y) - \Delta^2 w\right] - q + P_x w_{xx} + P_y w_{yy}\right\}\delta w\Bigg)dx\,dy + \\
&\oint\left\{\frac{Et}{1-\mu^2}\left[2(u_x + \mu v_y)\delta u\,dy - 2(v_y + \mu u_x)\delta v\,dx + (1-\mu)(u_y + v_x)(\delta v\,dy - \delta u\,dx)\right] + \right. \\
&\left.\frac{Et^3}{6(1-\mu^2)}\left[(w_{xx} + \mu w_{yy})\delta w_x\,dy - (w_{yy} + \mu w_{xx})\delta w_y\,dx\right]\right\} + \\
&2(1-\mu)\left[\left(w_{xy}\delta w\right)_{\substack{x=0\\y=0}} - \left(w_{xy}\delta w\right)_{\substack{x=L_x\\y=0}} + \left(w_{xy}\delta w\right)_{\substack{x=L_x\\y=L_y}} - \left(w_{xy}\delta w\right)_{\substack{x=0\\y=L_y}}\right] + \\
&\oint\left\{\frac{Et^3}{6(1-\mu^2)}\left[w_{yyy} + (2-\mu)w_{xxy}\right] + G_c c\left(\frac{2v}{c+t} - w_y\right) + P_y w_y\right\}\delta w\,dx - \\
&\oint\left\{\frac{Et^3}{6(1-\mu^2)}\left[w_{xxx} + (2-\mu)w_{xyy}\right] + G_c c\left(\frac{2u}{c+t} - w_x\right) + P_x w_x\right\}\delta w\,dy
\end{aligned}$$

where the meanings of the operators are

$$\left.\begin{aligned}\Delta^2 &= \left(\partial^2/\partial x^2\right) + \left(\partial^2/\partial y^2\right)\\ \Delta^4 &= \left(\partial^4/\partial x^4\right) + 2\left(\partial^4/\partial x^2\partial y^2\right) + \left(\partial^4/\partial y^4\right)\end{aligned}\right\} \tag{8a}$$

Since equation (8) must be satisfied identically for any arbitrary variation of the displacement functions u, v, and w, the following three differential equations must hold:

$$-\frac{Et}{1 - \mu^2}\left[2u_{xx} + (1 - \mu)u_{yy} + (1 + \mu)v_{xy}\right] + \frac{2G_c c}{c + t}\left[\left(\frac{2}{c + t}\right)u - w_x\right] = 0 \tag{9a}$$

$$-\frac{Et}{1 - \mu^2}\left[2v_{yy} + (1 - \mu)v_{xx} + (1 + \mu)u_{xy}\right] + \frac{2G_c c}{c + t}\left[\left(\frac{2}{c + t}\right)v - w_y\right] = 0 \tag{9b}$$

$$\frac{Et^3}{6\left(1 - \mu^2\right)}\Delta^4 w + G_c c\left[\left(\frac{2}{c + t}\right)\left(u_x + v_y\right) - \Delta^2 w\right] - q + P_x w_{xx} + P_y w_{yy} = 0 \tag{9c}$$

The line integrals in equation (8) furnish the boundary conditions. When all the four edges are simply supported the conditions prevailing along the edges can be represented by the equations

$$w = 0, \quad \delta w = 0, \quad \text{and} \quad \delta w_y = 0 \qquad \text{when} \qquad x = 0, L_x$$

$$w = 0, \quad \delta w = 0, \quad \text{and} \quad \delta w_x = 0 \qquad \text{when} \qquad y = 0, L_y$$

Hence equation (8) is satisfied identically if

$$u_x + \mu v_y = 0 \qquad \text{when} \qquad x = 0, L_x \tag{10a}$$

$$v_y + \mu u_x = 0 \qquad \text{when} \qquad y = 0, L_y \tag{10b}$$

$$u_y + v_x = 0 \qquad \text{when} \qquad x = 0, L_x \text{ and when } y = 0, L_y \tag{10c}$$

$$w_{xx} + \mu w_{yy} = 0 \qquad \text{when} \qquad x = 0,L_x \tag{10d}$$

$$w_{yy} + \mu w_{xx} = 0 \qquad \text{when} \qquad y = 0,L_y \tag{10e}$$

$$w = 0 \qquad \text{when} \qquad x = 0,L_x \quad \text{and when} \quad y = 0,L_y \tag{10f}$$

Equations (10a) and (10b) require that no moments be taken by the sandwich plate along the simply supported edges in the form of tension in the upper face and compression in the lower face, or vice versa. According to equations (10d) and (10e), the edges of the individual faces must also be free of bending moments. In equation (10c) the expression $u_y + v_x$ represents a shear strain in the upper face and an equal and opposite shear strain in the lower face. The two add up to a couple which is the torque applied to the edge of the plate. This must also vanish when the edges are simply supported.

When all the edges are rigidly clamped, the following edge conditions must be added to those listed earlier:

$$u = 0 \qquad \delta u = 0 \qquad w_x = 0 \qquad \delta w_x = 0$$

when $x = 0,L_x$, and

$$v = 0 \qquad \delta v = 0 \qquad w_y = 0 \qquad \delta w_y = 0$$

when $y = 0,L_y$. Hence with rigidly clamped edges the boundary conditions are

$$u = 0 \qquad \text{when} \qquad x = 0,L_x \tag{10g}$$

$$v = 0 \qquad \text{when} \qquad y = 0,L_y \tag{10h}$$

$$u_y + v_x = 0 \qquad \text{when} \qquad x = 0,L_x \quad \text{and when} \quad y = 0,L_y \tag{10i}$$

$$w = 0 \qquad \text{when} \qquad x = 0,L_x \quad \text{and when} \quad y = 0,L_y \tag{10j}$$

$$w_x = 0 \qquad \text{when} \qquad x = 0,L_x \tag{10k}$$

$$w_y = 0 \qquad \text{when} \qquad y = 0,L_y \tag{10l}$$

In many practical applications these boundary conditions can be relaxed slightly. It is usual to stiffen the edges of a sandwich panel by means of a rigid insert replacing the core if the core is weak in compression perpendicular to the faces. Without such an insert the reaction forces might damage the edges. When the edge is stiff enough, the torque corresponding to the shear strains represented by equations (10c) and (10i) can be replaced by statically equivalent couples consisting of shear stresses acting in the core perpendicular to the faces. These shears largely cancel one another and the noncanceling part, corresponding to the rate of change of the torque along the edge, can be equilibrated by distributed reaction forces along the edges and concentrated reaction forces at the four corners. Naturally rigid supports can always produce such reactions. Hence equations (10c) and (10i) need not be satisfied.

The replacement of the torque by the reactions is permissible from the standpoint of static equilibrium, but it causes incompatible deformations. Consequently the state of stress and strain must change in the plate. With sufficiently stiff edges these changes are negligibly small beyond a narrow band adjacent to the edges. This must be true by virtue of Saint Venant's principle. When the edges are weak, the band affected may be wide and thus equations (10c) and (10i) should not be disregarded. It is of interest to note that considerations similar to those just presented form an essential part of classical plate theory where they were introduced by Kirchhoff. They are described, for instance, on pages 47 and 89 of reference 8.

Another argument can also be advanced in favor of omitting equations (10c) and (10i). Simple supports in the case of sandwich plates may be construed to consist of individual knife edges along the edges of each face which permit the translation of the faces in a direction perpendicular to the edge of the panel but prevent translations along the edges. Consequently shear forces must be transmitted along the knife-edge supports which equilibrate the shear represented by the terms $u_y + v_x$. Under such conditions no v-displacements are possible along the edges parallel to the y-axis nor u-displacements along the edges parallel to the x-axis. Consequently δv is not a virtual displacement along the former edges and δu is not a virtual displacement along the latter. The line integral containing $u_y + v_x$ in equation (8) vanishes therefore automatically even though the shear stresses are finite. In practical design the two individual knife edges can be replaced by a single one and by a stiffening insert replacing the core along the edges of the sandwich plate with a consequent slight disturbance in the states of stress and strain in the neighborhood of the edges.

In view of these considerations, equations (10c) and (10i) can be replaced by the requirements of vanishing u- and v-displacements along the edges parallel to the x- and y-axes, respectively. The number of boundary conditions is thereby not altered. If it is observed that w_{xx} vanishes along the edges parallel to the x-axis and w_{yy} is zero along the edges parallel to the y-axis, the problem of the simply supported plate can be stated in the following alternative form:

$$D_o\left[2u_{xx} + (1 - \mu)u_{yy} + (1 + \mu)v_{xy}\right] - 2G_c cu + 2G_c c\,\frac{c + t}{2}\,w_x = 0 \qquad \text{(11a)}$$

$$D_o\left[2v_{yy} + (1 - \mu)v_{xx} + (1 + \mu)u_{xy}\right] - 2G_c cv + 2G_c c\,\frac{c + t}{2}\,w_y = 0 \qquad \text{(11b)}$$

$$D_f\,\Delta^4 w + G_c c\,\frac{2}{c + t}\left(u_x + v_y\right) - G_c c\,\Delta^2 w - q + P_x w_{xx} + P_y w_{yy} = 0 \qquad \text{(11c)}$$

$u_x + \mu v_y = 0$ when $x = 0, L_x$ (12a)

$v_y + \mu u_x = 0$ when $y = 0, L_y$ (12b)

$u = 0$ when $y = 0, L_y$ (12c)

$v = 0$ when $x = 0, L_x$ (12d)

$w_{xx} = 0$ when $x = 0, L_x$ (12e)

$w_{yy} = 0$ when $y = 0, L_y$ (12f)

$w = 0$ when $x = 0, L_x$ and when $y = 0, L_y$ (12g)

In equations (11) the symbols denoting bending rigidities are defined in the following manner:

$$D_o = Et(c + t)^2 \Big/ 2\left(1 - \mu^2\right) \qquad \text{(13a)}$$

$$D_f = Et^3 \Big/ 6\left(1 - \mu^2\right) \qquad \text{(13b)}$$

where D_f is the bending rigidity per inch of the faces about their own centroidal axes, calculated for the two faces, and D_o is the bending rigidity of a 1-inch-wide segment of the sandwich panel, calculated about the centroidal axis of the sandwich, neglecting the contribution of the core as well as that represented by D_f.

Finally the case of a free unsupported edge should be mentioned. When it is parallel to the x-axis, the boundary conditions are:

$$v_y + \mu u_x = 0 \tag{10m}$$

$$u_y + v_x = 0 \tag{10n}$$

$$w_{yy} + \mu w_{xx} = 0 \tag{10o}$$

$$\frac{Et^3}{6\left(1 - \mu^2\right)}\left[w_{yyy} + (2 - \mu)w_{xxy}\right] + G_c c\left(\frac{2v}{c + t} - w_y\right) + P_y w_y = 0 \tag{10p}$$

When the free edge is parallel to the y-axis, the boundary conditions become:

$$u_x + \mu v_y = 0 \tag{10q}$$

$$u_y + v_x = 0 \tag{10r}$$

$$w_{xx} + \mu w_{yy} = 0 \tag{10s}$$

$$\frac{Et^3}{6\left(1 - \mu^2\right)}\left[w_{xxx} + (2 - \mu)w_{xyy}\right] + G_c c\left(\frac{2u}{c + t} - w_x\right) + P_x w_x = 0 \tag{10t}$$

In equations (10p) and (10t) the bracketed term represents the distributed shear in the two faces augmented by the shear corresponding to the rate of change of the torque along the edges of the two faces. (See p. 90 of reference 8.) The next term is the shear carried by the core, and the last term is the component of the edge load perpendicular to the deflected surface of the faces. The equations represent, therefore, the condition that the resultant shear force must vanish along a free edge. The physical interpretation of the first three equations of each set was discussed earlier.

In addition the condition

$$w_{xy}\delta w = 0$$

must be fulfilled at each corner of the plate. When the corner is supported, the variation of the deflection vanishes. When the corner is unsupported, the deflection function must satisfy the requirement

$$w_{xy} = 0 \tag{10u}$$

BUCKLING OF A SIMPLY SUPPORTED SANDWICH PLATE

UNDER EDGEWISE COMPRESSION

When the compressive force is acting only in the y-direction and the transverse loading is absent, $P_x = q = 0$, the solution of the differential equations (11) and boundary conditions (12) can be written as

$$u = A \cos \frac{\pi x}{L_x} \sin \frac{n\pi y}{L_y} \tag{14a}$$

$$v = B \sin \frac{\pi x}{L_x} \cos \frac{n\pi y}{L_y} \tag{14b}$$

$$w = C \sin \frac{\pi x}{L_x} \sin \frac{n\pi y}{L_y} \tag{14c}$$

Substitution of these functions in equations (11) yields three homogeneous linear equations whose determinant must vanish when buckling occurs. The determinant is:

$$\begin{vmatrix} P_o\left[2 + (1-\mu)K^2\right] + 2G_c c & (1+\mu)KP_o & -2G_c c\frac{c+t}{2}\left(\frac{\pi}{L_x}\right) \\ (1+\mu)KP_o & P_o\left[2K^2 + (1-\mu)\right] + 2G_c c & -2G_c c\frac{c+t}{2}\left(\frac{\pi}{L_x}\right)K \\ -G_c c\,\frac{2}{c+t}\left(\frac{\pi}{L_x}\right) & -G_c c\,\frac{2}{c+t}\left(\frac{\pi}{L_x}\right)K & \left(\frac{\pi}{L_x}\right)^2\left[P_f\left(1+K^2\right)^2 + G_c c\left(1+K^2\right) - P_y K^2\right] \end{vmatrix} \tag{15}$$

where

$$\left.\begin{aligned} P_o &= \left(\frac{\pi}{L_x}\right)^2 D_o = \frac{\pi^2 Et(c + t)^2}{2\left(1 - \mu^2\right)L_x^2} \\ P_f &= \left(\frac{\pi}{L_x}\right)^2 D_f = \frac{\pi^2 Et^3}{6\left(1 - \mu^2\right)L_x^2} \\ K &= n\frac{L_x}{L_y} \end{aligned}\right\} \tag{16}$$

The symbol K designates the side ratio of the bulge, and P_o and P_f are proportional to the buckling loads of ordinary plates having bending rigidities D_o and D_f, respectively.

Expansion of the determinant and solution for $P_y = P_{cr}$ yield the buckling load

$$P_{cr} = \frac{\left(1 + K^2\right)}{K^2}\left[\left(1 + K^2\right)P_f + \frac{G_c c\left(1 + K^2\right)P_o}{G_c c + \left(1 + K^2\right)P_o}\right] \tag{17}$$

TRANSFORMATION OF DIFFERENTIAL EQUATIONS

If equation (11a) is differentiated with respect to x, and equation (11b) with respect to y, and the resulting equations are added, one obtains

$$\left(1 - \frac{D_o}{G_c c}\Delta^2\right)\left(u_x + v_y\right) = \frac{c + t}{2}\Delta^2 w \tag{18}$$

Equation (11c) can now be solved for $\left(u_x + v_y\right)$ and the operator $\left[1 - \left(D_o/G_c c\right)\Delta^2\right]$ applied to the resulting equations. After manipulations the following sixth-order partial differential equation is obtained:

$$D_f\,\Delta^6 w - \left(D_o + D_f\right)\left(G_c c/D_o\right)\Delta^4 w = \left[\Delta^2 - \left(G_c c/D_o\right)\right]\left(q - P_x w_{xx} - P_y w_{yy}\right) \tag{19}$$

where

$$\Delta^6 = \left(\partial^6/\partial x^6\right) + 3\left(\partial^6/\partial x^4\partial y^2\right) + 3\left(\partial^6/\partial x^2\partial y^4\right) + \left(\partial^6/\partial y^6\right) \qquad (19a)$$

Assumption of w in the form given in equation (14c) again yields the expression in the right-hand member of equation (17) for the buckling load.

It is of interest to note that equation (19) reduces to the differential equation of the sandwich plate given by Reissner as equation (71) in reference 6 if D_f is assumed to be zero.

EVALUATION OF BUCKLING FORMULA

For any side ratio L_y/L_x the buckling load P_{cr} can be calculated from equation (17) if n is assumed as some positive integer. Different assumptions yield different buckling loads, and buckling occurs according to the pattern whose buckling load is the smallest. Figure 5 shows the variation of the buckling load with L_y/L_x for several values of the ratio c/t. The symbol λ is defined as

$$P_{cr}/P_f = \sigma_{cr}/\sigma_{cr,f} = \lambda \qquad (20)$$

It can be seen from the diagram that the buckling stress varies little with the side ratio L_y/L_x. For $c/t = 25$ and $\bar{R} = 2G_c t/P_f = G_c/\sigma_{cr,f} = 131.7$, the buckling stress is practically constant for all values of L_y/L_x greater than 0.2. When $c/t = 7.5$ and $\bar{R} = 131.7$ the limit above which the buckling stress is constant is 0.6. As a matter of fact, the limit is always less than 1 except when the core is infinitely rigid in shear. Then the limit is 1 in agreement with thin-plate theory. For this reason the side ratio is not a parameter of major importance and the minimum buckling stress of all rectangular sandwich panels can be presented in a single diagram.

The dashed line near the left edge of the diagram is the boundary to the right of which the side ratio has a negligible effect upon the buckling load. The values of the numerical factor λ corresponding to the buckling load of these "long" plates are plotted as the dash-dotted line near the right edge of the diagram. The ordinates of this curve are the values of λ, and the abscissas the corresponding values of the thickness ratio c/t.

Data obtained from many similar diagrams prepared for various values of $\bar{R}$ are collected in figure 6. The abscissa is the thickness ratio c/t and the ordinate is a numerical factor C defined as

$$(1/4)P_{cr}\Big/\left(P_o + P_f\right) = C \tag{21}$$

The critical stress can be calculated from the formula

$$\sigma_{cr} = CF\sigma_{cr,f} \tag{22}$$

where $\sigma_{cr,f}$ would be the buckling stress of the two faces if they were not connected with each other by means of the core:

$$\sigma_{cr,f} = \frac{\pi^2 E t^2}{3\left(1 - \mu^2\right){L_x}^2} \tag{23}$$

and F is the form factor:

$$F = 1 + 3\left(1 + \frac{c}{t}\right)^2 \tag{24}$$

Since FD_f is the total bending rigidity of a sandwich panel whose core does not carry bending stresses but is infinitely rigid in shear, $F\sigma_{cr,f}$ is the buckling stress of a sandwich panel that is not subject to shearing deformations, and C is a reduction factor for the buckling stress of the actual sandwich panel. The value of C depends on the sandwich buckling parameter

$$R = G_c\Big/\left(F\sigma_{cr,f}\right) \tag{25}$$

In figure 6, R is the parameter of the family of curves.

Figure 7 presents the same information as figure 6 but the scale used for c/t is logarithmic to facilitate reading the values of C in the region $0.5 < \lambda < 1.5$.

NUMERICAL EXAMPLE

As an example of the application of the buckling formula just derived, the buckling stress of an alclad-balsa sandwich plate will now be calculated. The plate is square with a side length of 23.5 inches. Other pertinent data are:

$$t = 0.021 \text{ inch} \qquad c = 0.181 \text{ inch}$$

$$G_c = 19{,}000 \text{ psi} \qquad E = 9.5 \times 10^6 \text{ psi}$$

$$\mu = 0.3$$

According to equation (24),

$$F = 1 + 3(1 + 8.62)^2 = 278$$

From equation (23),

$$\sigma_{cr,f} = \frac{\pi^2 9.5 \times 10^6 (0.021)^2}{3 \times 0.91 \times 23.5^2} = 27.25 \text{ psi}$$

Consequently,

$$F\sigma_{cr,f} = 7560 \text{ psi}$$

From equation (25),

$$R = 19{,}000/7560 = 2.51$$

The value of C can now be found from figure 7 for $c/t = 8.62$ and $R = 2.51$:

$$C = 0.955$$

The critical stress can be computed from equation (22):

$$\sigma_{cr} = 0.955 \times 7560 = 7220 \text{ psi}$$

The critical load per inch of the width of the panel is therefore

$$P_{cr} = 0.042 \times 7220 = 303 \text{ lb/in.}$$

The four specimens of this type tested at the Forest Products Laboratory and reported in table 3 of reference 9 failed under loads ranging from 266 to 300 psi. This, as well as other similar comparisons, indicates that there was good agreement between the theory of the present investigation and results of tests carried out at the Forest Products Laboratory.

If the balsa core is replaced by a cellular cellulose acetate core having a shear modulus

$$G_c = 2500 \text{ psi}$$

the ratio R becomes

$$R = 2500/7560 = 0.331$$

and thus figure 7 yields

$$\bar{C} = 0.725$$

Hence

$$\sigma_{cr} = 0.726 \times 7560 = 5475 \text{ psi}$$

and

$$P_{cr} = 0.042 \times 5475 = 230 \text{ lb/in.}$$

CONCLUDING REMARKS

Differential equations have been developed for the calculation of the deflections and the buckling load of rectangular sandwich panels subjected to transverse loads and edgewise compression. The equations have been solved for simply supported panels compressed parallel to one pair of edges. The results of the calculations are presented in a diagram which permits a rapid computation of the buckling loads. Good agreement was obtained with results of tests carried out at the Forest Products Laboratory.

Polytechnic Institute of Brooklyn
Brooklyn, N. Y., May 27, 1949

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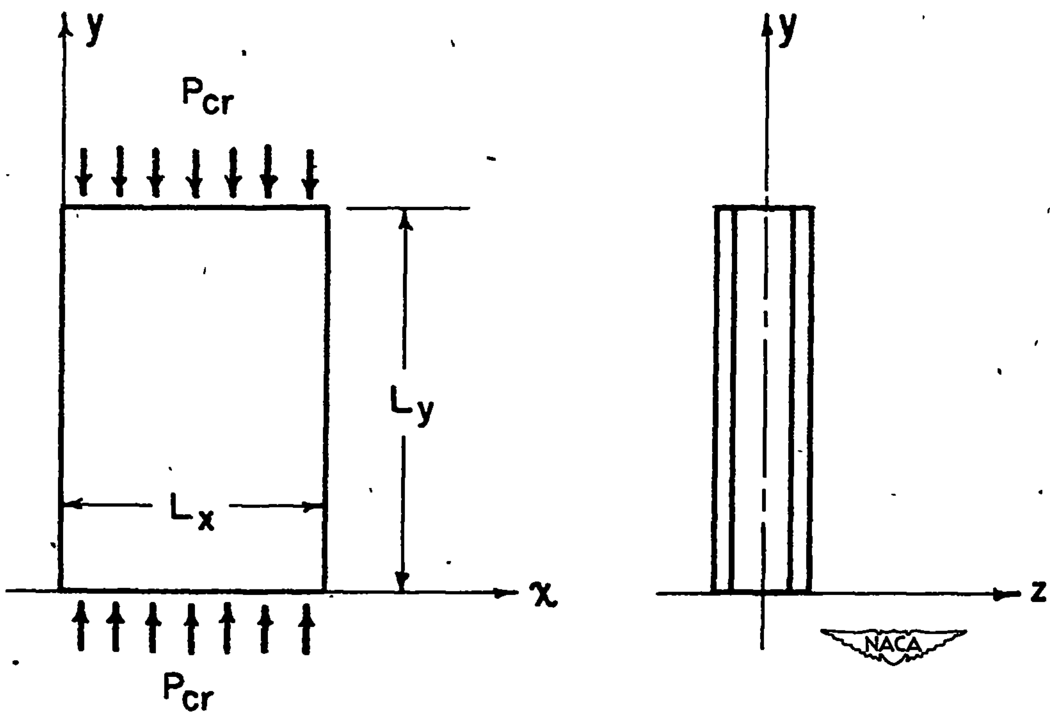

Figure 1.- Sketch of sandwich plate with thicknesses exaggerated.

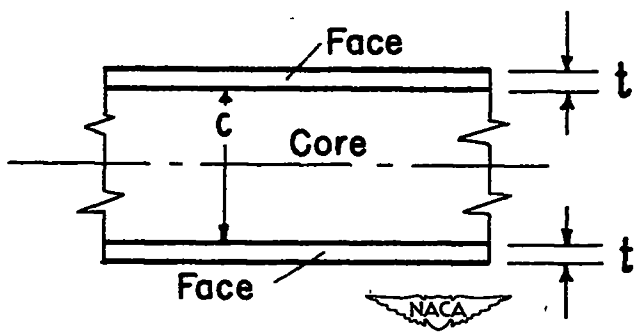

Figure 2.- Section of sandwich plate.

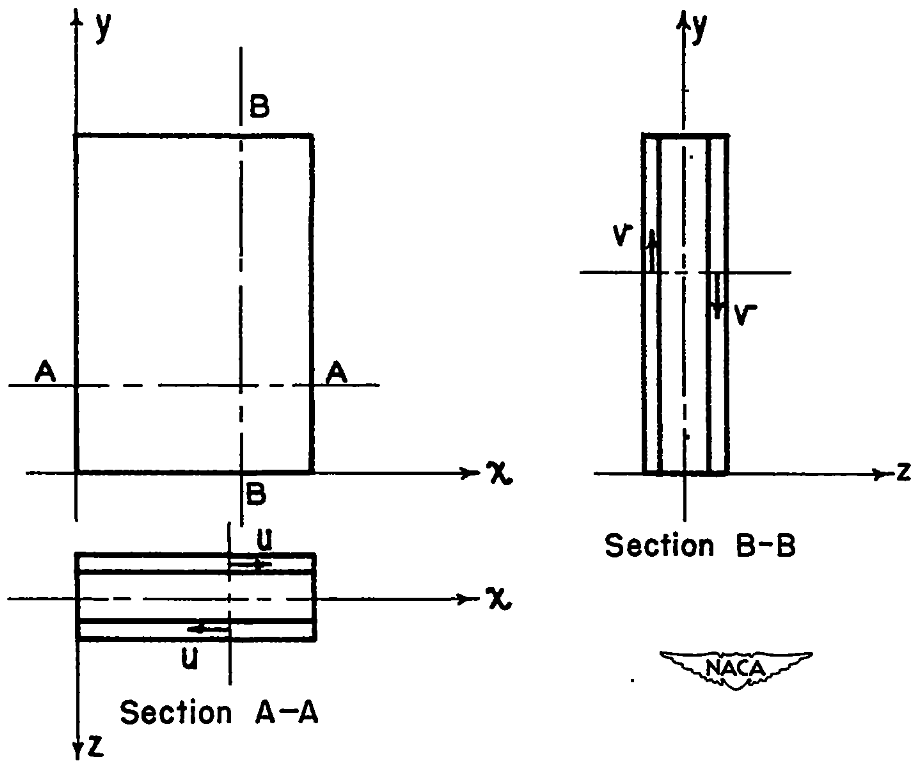

Figure 3.- Displacements in plane of plate with thicknesses exaggerated.

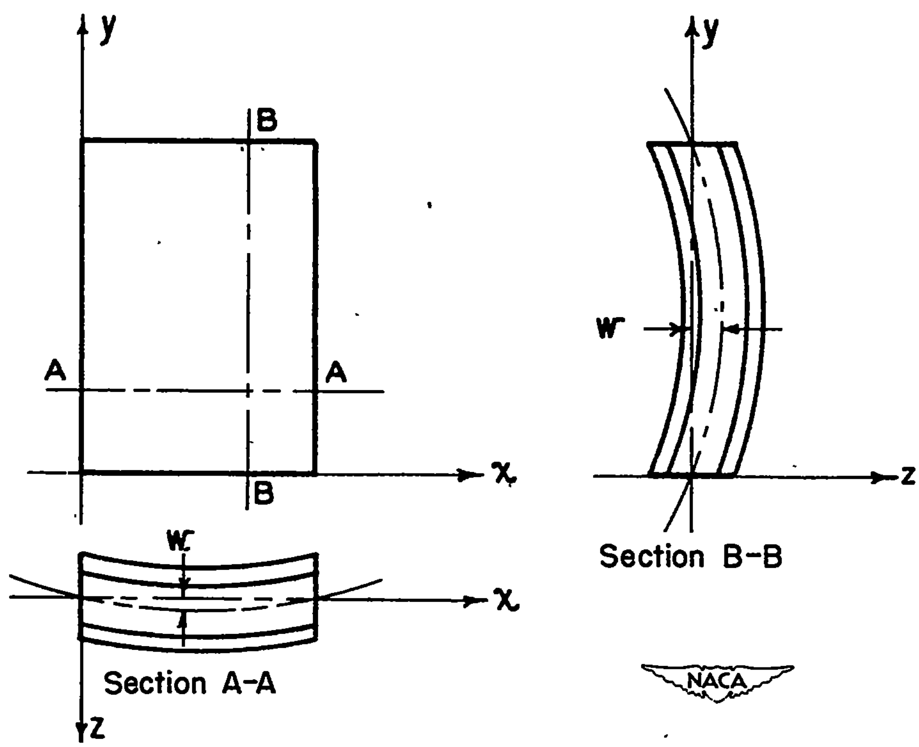

Figure 4.- Displacements out of plane of plate with thicknesses exaggerated.

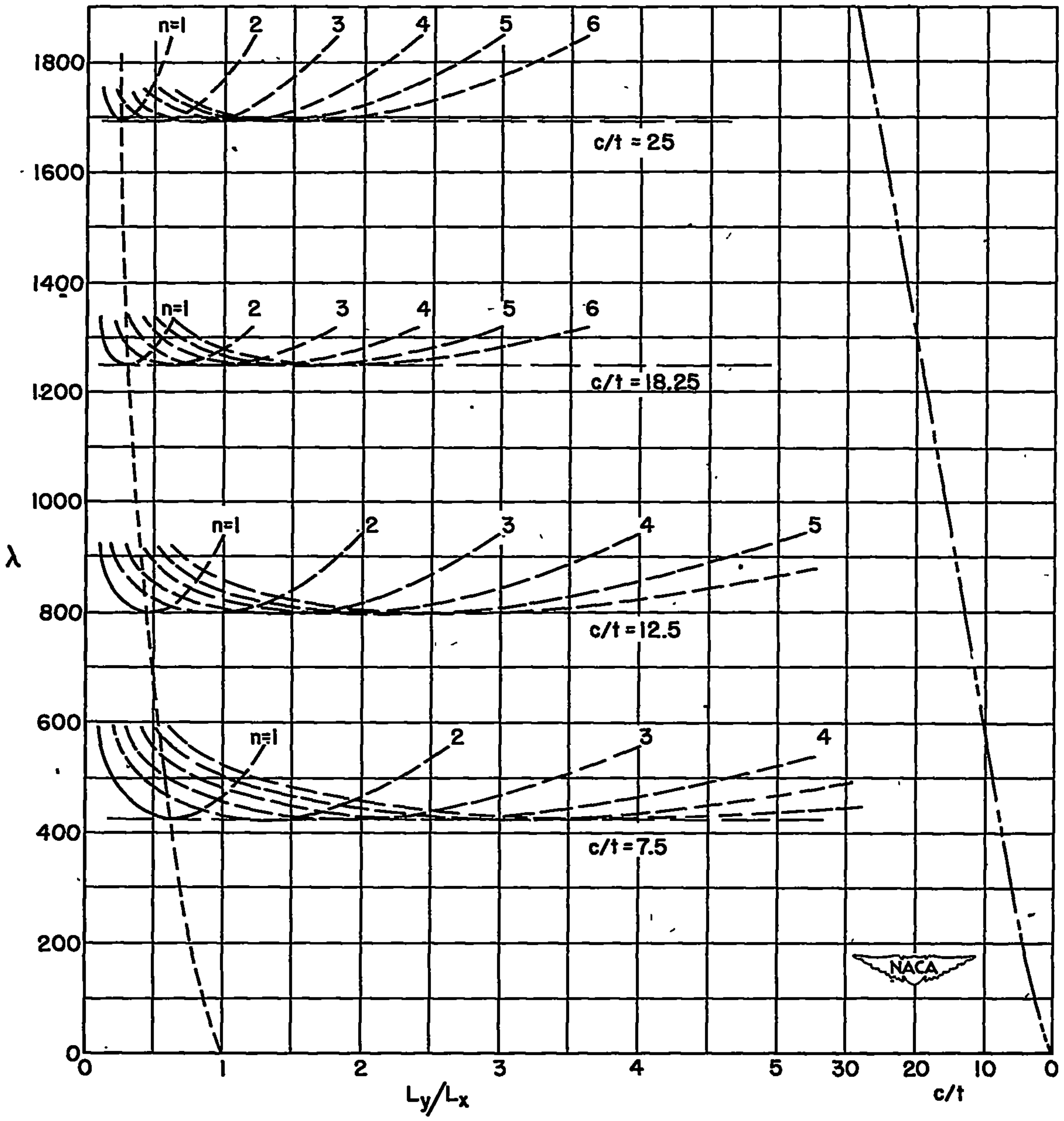

Figure 5.- Buckling stress as a function of side ratio and thickness ratio. $\bar{R}$ = 131.717. Dashed line near left edge of diagram is boundary to right of which side ratio has negligible effect on buckling load.

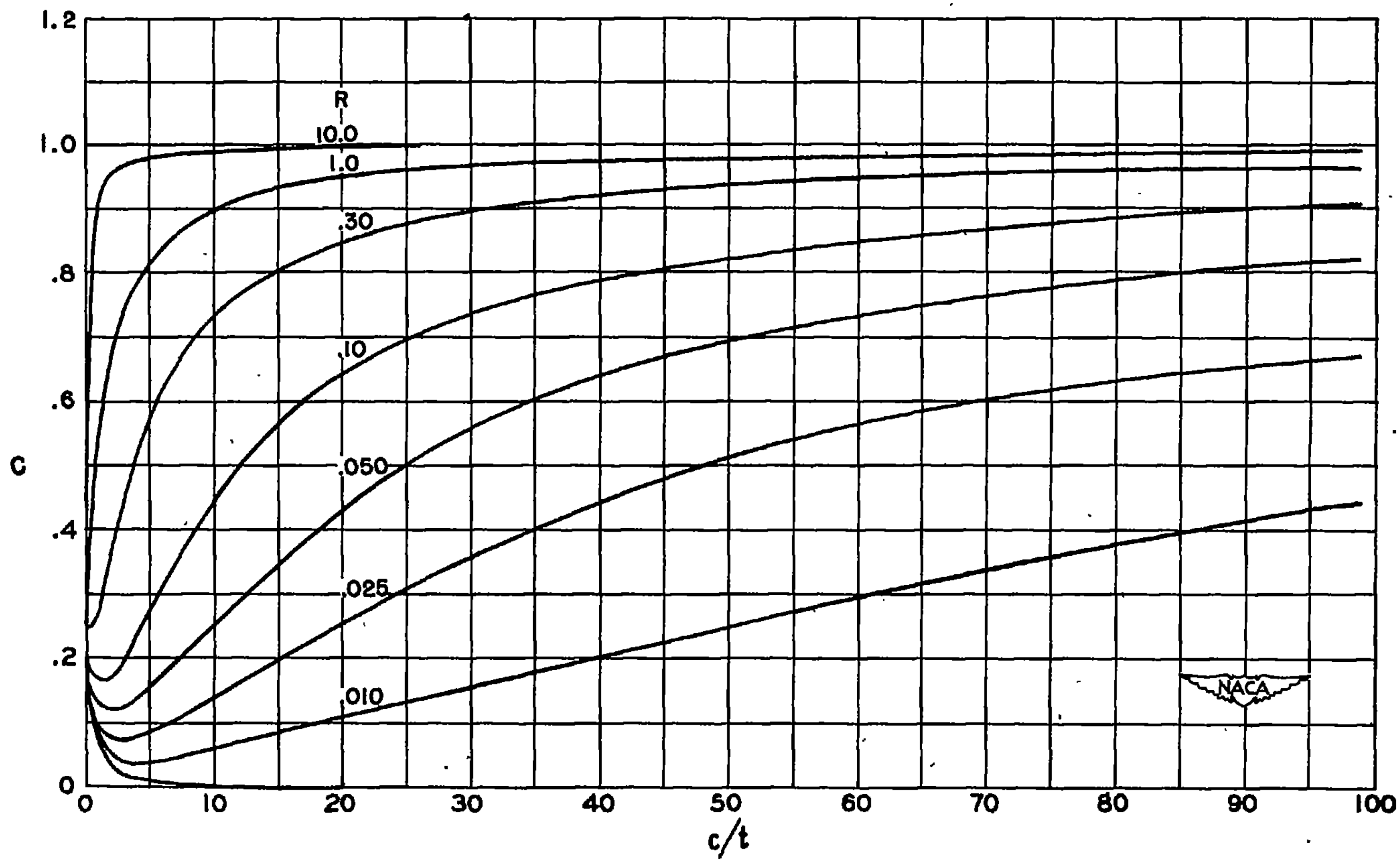

Figure 6.- Buckling stress of simply supported rectangular sandwich panels.

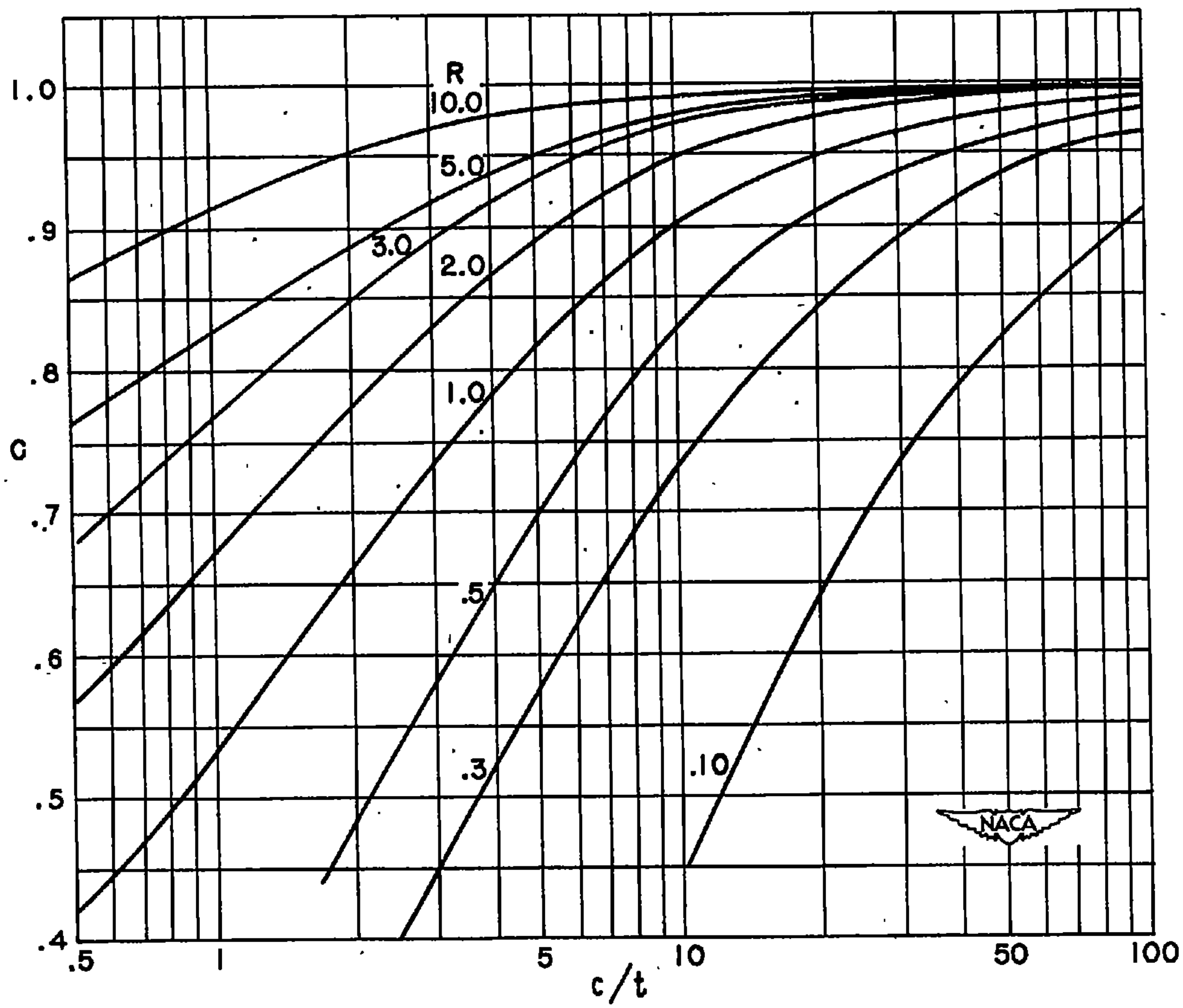

Figure 7.- Logarithmic buckling-stress diagram for simply supported rectangular sandwich panels.

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